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Noncommutative Maxwell–Chern–Simons theory, duality and a new noncommutative Chern–Simons theory in d=3

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Abstract

Noncommutative Maxwell–Chern–Simons theory in 3–dimensions is defined in terms of star product and noncommutative fields. Seiberg–Witten map is employed to write it in terms of ordinary fields. A parent action is introduced and the dual action is derived. For spatial noncommutativity it is studied up to second order in the noncommutativity parameter θ. A new noncommutative Chern–Simons action is defined in terms of ordinary fields, inspired by the dual action. Moreover, a transformation between noncommuting and ordinary fields is proposed.

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1 Introduction

An equivalence of "ordinary" (commutative) and noncommutative gauge fields leads to a transformation between them which is known as Seiberg–Witten (SW) map[1]. This permits one to study noncommutative gauge theories in terms of ordinary fields. In fact, in [2] (S) duality is incorporated into noncommutative Maxwell theory action in terms of ordinary fields after performing SW map. In 4–dimensions if the original theory is noncommutative Maxwell theory where noncommutativity is spatial, its dual theory is a noncommutative gauge theory whose time variable is effectively noncommuting with the other coordinates[2]. This interesting phenomenon is a consequence of the fact that the duality transformation includes 4–dimensional totally antisymmetric tensor.

In 3–dimensions the most extensively studied duality is between Maxwell–Chern–Simons (MCS) theory and self dual theory[3]. It leads to two equivalent descriptions of the dynamics of massive spin–1 field. One of its most known applications is bosonization in 3–dimensions[4]. We wonder what would be the consequences of generalization of this duality to noncommutative MCS theory. In [5] and [6] some generalizations of the mentioned duality to noncommutative theories are investigated in terms of noncommuting fields. However, duality can also be studied employing ordinary fields in the spirit of [2]. Although at first sight this can appear to be trivial due to the fact that 3–dimensional noncommutative Chern–Simons (CS) action becomes the usual CS action in terms of SW map[7]–[10], we will show that it gives nontrivial results.

We write 3–dimensional noncommutative MCS action in terms of ordinary gauge fields utilizing SW map. We introduce a parent action in terms of ordinary fields to obtain the dual description. We study the dual action up to the second order in the noncommutativity parameter θ, when we let only spatial noncommutativity. Once the dual description is obtained it inspires a new noncommutative CS theory in terms of ordinary gauge fields. We discuss equations of motion following from this new action. Moreover, we propose to write it in terms of noncommuting fields as the simplest generalization of abelian CS action. This leads to an explicit transformation between noncommutative and ordinary fields.

2 Duality and Noncommutative MCS Theory

It is well known that noncommutativity between coordinates can be introduced in terms of the star product

$$* \equiv \exp \frac{i\theta^{\mu\nu}}{2}\left(\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu - \overleftarrow{\partial}_\nu \overrightarrow{\partial}_\mu\right), \tag{1}$$

where $\theta^{\mu\nu}$ are antisymmetric constant parameters. Thus, one retains coordinates commuting under the usual product.

Noncommutative MCS theory in 3–dimensions can be defined as

$$S = \hat{S}_M + \hat{S}_{CS}, \tag{2}$$

in terms of the noncommutative CS action

$$\hat{S}_{CS} = \frac{m}{2}\epsilon^{\mu\nu\rho}\int d^3x \left(\hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{2}{3}\hat{A}_\mu * \hat{A}_\nu * \hat{A}_\rho\right) \tag{3}$$

and the noncommutative Maxwell theory

$$\hat{S}_M = -\frac{1}{4}\int d^3x \hat{F}_{\mu\nu}\hat{F}^{\mu\nu}. \tag{4}$$

We employed the noncommutative field strength:

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu. \tag{5}$$

$\hat{A}^\mu$ are not operators but they are called noncommutative gauge fields in the sense that they take values in noncommutative space.

One can show that the noncommutative MCS action (2) is invariant under the gauge transformations

$$\hat{\delta}_{\hat{\lambda}}\hat{A}^\mu = \partial^\mu \hat{\lambda} + i\hat{\lambda} * \hat{A}^\mu - i\hat{A}^\mu * \hat{\lambda}. \tag{6}$$

The equivalence relation between the noncommuting $\hat{A}^\mu,\ \hat{\lambda}$ and the ordinary (commuting) gauge fields and gauge parameter $A^\mu,\ \lambda:$

$$\hat{A}^\mu(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}^\mu(A) = \hat{A}^\mu(A + \delta_\lambda A), \tag{7}$$

leads to the SW map[1]. To the first order in θ it is written explicitly as

$$\hat{A}^\mu = A^\mu - \theta^{\rho\nu}(A_\rho \partial_\nu A_\mu - \frac{1}{2}A_\rho \partial_\mu A_\nu). \tag{8}$$

When the change of variables which follows from (7) is performed the noncommutative CS action (3) becomes the usual action[7]

$$\hat{S}_{CS} = \frac{m}{2}\epsilon^{\mu\nu\rho}\int d^3x A_\mu \partial_\nu A_\rho. \tag{9}$$

Thus, in terms of the SW map the action (2) can be expressed as

$$S = \int d^3x \left\{ -\frac{1}{4}\left[F_{\mu\nu}F^{\mu\nu} + \mathcal{L}(\theta, F)\right] + \frac{m}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho \right\}, \tag{10}$$

where the θ dependent part can be written as

$$\mathcal{L}(\theta, F) \equiv L_\theta(F) + L_{\theta^2}(F) + \cdots.$$

L_{θ^n} is at the nth order in θ. In the following we will use only the first and the second order terms, which can be written as[2]

$$\begin{aligned} L_\theta(F) &= 2\theta_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - 12\theta_{\mu\nu}F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, & (11)\\ L_{\theta^2}(F) &= 2\theta_{\nu\mu}F^{\nu\rho}\theta_{\rho\sigma}F^{\sigma\delta}F_{\delta\xi}F^{\xi\mu} + \theta_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}\theta^{\sigma\delta}F_{\delta\xi}F^{\xi\mu} + \theta_{\mu\nu}F^{\nu\mu}\theta_{\rho\sigma}F^{\sigma\xi}F_{\xi\delta}F^{\delta\rho} \\ &\quad -\frac{1}{8}(\theta_{\mu\nu}F^{\mu\nu})^2 F_{\rho\sigma}F^{\sigma\rho} + \frac{1}{4}\theta_{\mu\nu}F^{\nu\rho}\theta_{\rho\sigma}F^{\sigma\mu}F_{\delta\xi}F^{\xi\delta}. & (12) \end{aligned}$$

Let us introduce the parent action

$$\begin{aligned} S &= \int d^3x \Big\{ -\epsilon^{\mu\nu\rho}B_\mu \partial_\nu A_\rho + \frac{1}{2}B_\mu B^\mu + \frac{m}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho \\ &\quad -\frac{1}{4}\left[L_\theta(F) + L_{\theta^2}(F) + \cdots\right] \Big\}. \end{aligned} \tag{13}$$

Equations of motion with respect to B_μ are

$$B_\mu = \epsilon_{\mu\nu\rho}\partial^\nu A^\rho.$$

When we substitute B^μ with this in (13) the noncommutative MCS action (10) follows.

On the other hand the equations of motion with respect to A_μ

$$\partial_\nu \left[\epsilon^{\mu\nu\rho}(B_\rho - mA_\rho) - \frac{1}{2}\frac{\delta\mathcal{L}}{\delta F_{\nu\mu}} \right] = 0, \tag{14}$$

can be solved for A^μ as

$$A_\mu = \frac{1}{m}B_\mu + \frac{1}{m}b_\mu(\theta, B). \tag{15}$$

We defined $b_\mu(\theta, B)$ in terms of the equation

$$b_\mu(\theta, B) + \frac{1}{4}\epsilon_{\mu\nu\rho}\mathcal{K}_\theta^{\nu\rho}(\frac{H}{m} + \frac{h(\theta, B)}{m}) = 0, \tag{16}$$

where $H = dB,\ h(\theta, B) = db(\theta, B)$ and

$$\mathcal{K}_\theta^{\mu\nu}(F) \equiv \frac{\delta\mathcal{L}(\theta, F)}{\delta F_{\mu\nu}}.$$

Obviously, $b_\mu(\theta, B)$ can be expanded in powers of θ as

$$b^\mu(\theta, B) = b_\theta^\mu + b_{\theta^2}^\mu + \cdots.$$

When we plug the solution (15) into (13) the dual of noncommutative MCS action follows:

$$\begin{aligned} S_D \quad = \int d^3x \quad & \Big\{ \frac{1}{2}B_\mu B^\mu + \frac{1}{2m}\epsilon_{\mu\nu\rho}\left[B^\mu\partial^\nu B^\rho + b^\mu(\theta, B)\partial^\nu b^\rho(\theta, B)\right] \\ & -\frac{1}{4}\mathcal{L}\left(\theta, \frac{H}{m} + \frac{h(\theta, B)}{m}\right) \Big\}. \end{aligned} \tag{17}$$

In the most general case we can add to the solution (15) the term $\partial_\mu\kappa$, where κ is an arbitrary function. However, this alters the dual action (17) only up to a surface term which we can drop. Obviously, this is equivalent to choose a vanishing κ.

When the noncommutativity is along the spatial coordinates:

$$\theta^{ij} = \theta\epsilon^{ij},\ \theta^{0i} = 0, \tag{18}$$

the first order term (11) can be written as

$$L_\theta(F) = \theta F_{12} F_{\mu\nu} F^{\mu\nu}. \tag{19}$$

Now, we can solve (16) to obtain

$$\begin{aligned} b_\theta^0 &= \frac{\theta}{m^2}\left[H_{\mu\nu}H^{\mu\nu} + 2H_{12}H_{12}\right], \\ b_\theta^1 &= \frac{2\theta}{m^2}H_{12}H_{02}, \\ b_\theta^2 &= \frac{2\theta}{m^2}H_{12}H_{10}. \end{aligned} \tag{20}$$

When we use these in (17) explicit form of the dual action to the second order in θ follows. To the second order in θ (17) can be written as

$$\begin{aligned} S_{D,(2)} \equiv \int d^3x \quad & \left\{ \frac{1}{2} B_\mu B^\mu + \frac{1}{2m} \epsilon_{\mu\nu\rho} B^\mu \partial^\nu B^\rho - \frac{1}{4} L_\theta \left(\frac{H}{m} \right) + \frac{1}{2m} \epsilon_{\mu\nu\rho} b_\theta^\mu \partial^\nu b_\theta^\rho \right. \\ & - \frac{1}{4} L_{\theta^2} \left(\frac{H}{m} \right) + \frac{2}{m^2} \theta_{\mu\nu} \left(H^{\nu\rho} h_{\theta\rho\sigma} H^{\sigma\mu} + 2 H^{\nu\rho} H_{\rho\sigma} h_\theta^{\sigma\mu} \right) \\ & \left. - \frac{12}{m^2} \theta_{\mu\nu} \left(h_\theta^{\mu\nu} H_{\rho\sigma} H^{\rho\sigma} + 2 H^{\mu\nu} h_{\theta\rho\sigma} H^{\rho\sigma} \right) \right\}, \end{aligned}$$

where $h_\theta^{\mu\nu} = \partial^\mu b_\theta^\nu - \partial^\nu b_\theta^\mu$.

Although, the dual actions (10) and (17) are obtained from the parent action (13), it is not guaranteed that they yield the same partition function. Indeed, we deal only with the classical aspects. Quantum corrections may oblige us to regulate the action (13) [2], which is not addressed in this work.

3 A New Noncommutative CS Theory

When SW map is employed the noncommutative CS action (3) becomes the ordinary CS action (9) [7]. Thus, a noncommutative CS theory formulated in terms of the ordinary gauge fields A_μ and the noncommutativity parameter θ is not available. However, we can utilize the action S_D (17) to define a new noncommutative abelian CS theory in terms of the ordinary gauge fields B^μ and the noncommutativity parameter $\theta_{\mu\nu}$:

Setting $\theta^{\mu\nu} = 0$ and dropping the ordinary mass term B^2 in S_D (17) lead to the ordinary abelian CS action:

$$S_{CS}[B] = \frac{M}{2} \int d^3x \; \epsilon_{\mu\nu\rho} B^\mu \partial^\nu B^\rho, \tag{21}$$

where $M \equiv 1/m$. We would like to take advantage of this observation to define a new noncommutative abelian CS theory in terms of the ordinary gauge fields B^μ as

$$S_{\rm NCS} = \int d^3x \left\{ \frac{M}{2} \epsilon_{\mu\nu\rho} \left[B^\mu \partial^\nu B^\rho + b^\mu(\theta, B) \partial^\nu b^\rho(\theta, B) \right] - \frac{1}{4} \mathcal{L} \left(\theta, MH + Mh(\theta, B) \right) \right\}, \tag{22}$$

by dropping the B^2 term in (17). Obviously, this action is invariant under the abelian gauge transformations $\delta B_\mu = \partial_\mu \lambda$ and yields the ordinary CS theory (21) when one sets $\theta = 0$.

Equations of motion are

$$\epsilon_{\mu\nu\rho} \partial^\nu \left(B^\rho + b^\rho(\theta, B)) \right) - 4 \epsilon_{\sigma\nu\rho} \partial^\kappa \left[G^\sigma_{\kappa\mu}(H) \partial^\nu b^\rho \right] = 0, \tag{23}$$

where we defined

$$\frac{\delta b^\mu(\theta, B(y))}{\delta H^{\nu\rho}(x)} = G^\mu_{\nu\rho}(H) \delta^3(x-y). \tag{24}$$

Observe that the simplest solution of (23) is

$$H^{\mu\nu} = 0,$$

which is independent of θ. To the first order in θ the equations of motion (23) get the simple form

$$\epsilon_{\mu\nu\rho}\partial^\nu (B^\rho + b_\theta^\rho) = 0. \tag{25}$$

The SW map (7) expresses the noncommutative gauge fields $\hat{A}_\mu$ in terms of the ordinary gauge fields A_μ utilizing the equivalence relation (8). A transformation between noncommutative and ordinary fields can also be derived by assuming an equivalence relation between the action (22) and another one written by introducing some fields $\mathcal{B}(B,\theta)$ taking values in noncommutative space. However, there is no unique choice for the latter action. One should make an assumption about the form of the action in terms of the noncommuting fields $\mathcal{B}(B,\theta)$. Let us suppose that the action in terms of the noncommutative fields $\mathcal{B}(B,\theta)$, is in the same form as the abelian CS theory:

$$S_{NCS} \equiv \frac{M}{2}\int d^3x \epsilon_{\mu\nu\rho}\mathcal{B}^\mu(B,\theta)\partial^\nu\mathcal{B}^\rho(B,\theta). \tag{26}$$

One can show that there exists a transformation between $\mathcal{B}^\mu(B,\theta)$ and B^μ. Indeed, one can solve for $\mathcal{B}$ perturbatively in θ. At the first order in θ one should solve

$$\epsilon_{\mu\nu\rho}B_\theta^\mu H^{\rho\nu} = L_\theta(MH), \tag{27}$$

where $B_\theta^\mu = \partial\mathcal{B}^\mu/\partial\theta|_{\theta=0}$. There is not a unique solution. For instance, when the noncommutativity is only spatial (18), a solution is

$$B_\theta^\mu = \frac{M\theta}{2}H_{21}\epsilon^{\mu\nu\rho}H_{\nu\rho}.$$

Although the assumption (26) is very plausible, in principle one may define some other actions in terms of fields taking values in noncommutative space. Nevertheless, the assumed form of the action (26) is shown to yield a map between the noncommutative gauge fields $\mathcal{B}^\mu(B,\theta)$ and the ordinary ones B^μ which is not the SW map (8). Moreover, the form of the action (26) can be useful to generalize this construction to nonabelian gauge theories.

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